Math 255B Lecture 15 Notes

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1 The Kato-Rellich Theorem

1.1 The Kato-Rellich theorem

Last time, we were in the middle of proving the Kato-Rellich theorem.

Theorem 1.1 (Kato-Rellich). *Let A be self-adjoint, and let B be symmetric and A-bounded with relative bound < 1. Then $A + B$ is self-adjoint on $D(A)$.*

Proof. $A+B$ is closed, symmetric, and densely defined on $D(A)$. So we only need to show that the deficiency indices are 0: that is, we want $\operatorname{Im}(A + B \pm i) = H$. In fact, we will show that there exists some $\lambda \in \mathbb{R} \setminus \{0\}$ such that $\operatorname{Im}(A + B \pm i\lambda) = H$.

As A is self-adjoint, this is true when $B = 0$. We have

$$\|(A + i\lambda)u\|^2 = \|Au\|^2 + \lambda^2\|u\|^2 \qquad \forall u \in D(A).$$

So we know that $A + i\lambda : D(A) \to H$ is bijective, and

$$\|v\|^2 = \|A(A + i\lambda)^{-1}v\|^2 + \lambda^2\|(A + i\lambda)^{-1}v\|^2 \qquad \forall v \in H.$$

So $(A + i\lambda)^{-1}, A(A + i\lambda)^{-1} \in \mathcal{L}(H, H)$ with

$$\|(A + i\lambda)^{-1}\| \leq \frac{1}{|\lambda|}, \qquad \|A(A + i\lambda)^{-1}\| \leq 1.$$

Next by the A-boundedness of B, there exists some $0 \leq a < 1$ such that for any $u \in H$,

$$\begin{aligned}
\|B(A + i\lambda)^{-1}u\| &\leq a\|A(A + i\lambda)^{-1}u\| + b\|(A + i\lambda)^{-1}u\| \\
&\leq a\|u\| + \frac{b}{|\lambda|}\|u\| \\
&= \left(a + \frac{b}{|\lambda|}\right)\|u\|
\end{aligned}$$

Pick λ large enough to get

$$= \left(\frac{1+a}{2}\right)\|u\|.$$

Thus, the operator $1 + B(A+i\lambda)^{-1}$ is invertible in $\mathcal{L}(H,H)$. We get that

$$A + B + i\lambda = (1 + B(A+i\lambda)^{-1})(A+i\lambda) : D(A) \to H$$

is bijective. So $A + B$ is self-adjoint on $D(A)$. $\square$

Here is an application:

Example 1.1 (Schrödinger operator with a Coulomb potential)**.** Let $H = L^2(\mathbb{R}^3)$, and let $P_0 = -\Delta$ (self-adjoint with $D(P_0) = H^2(\mathbb{R}^3)$). Our potential is $V(x) = \frac{\gamma}{|x|}$ with $\gamma \in \mathbb{R}$.

We claim that $P = P_0 + V$ is self-adjoint on L^2 with domain $D(P) = H^2$. We may assume that $|\gamma|$ is small, for we can change scales: Introduce $U_\lambda : L^2 \to L^2$ which acts as $(U_\lambda f)(x) = \lambda^{-n/2} f(x/\lambda)$. Then

$$U_\lambda^{-1}(-\Delta + V)U_\lambda = -\frac{1}{\lambda^2}\Delta + V(\lambda\,\cdot) = -\frac{1}{\lambda^2}\Delta + \frac{\gamma}{\lambda|x|} = \frac{1}{\lambda^2}\left(-\Delta + \frac{\lambda\gamma}{|x|}\right).$$

So we don't need to worry so much about the relative bound in the Kato-Rellich theorem.

We shall show that

$$\|Vu\|^2 = \int \frac{|u(x)|^2}{|x|^2}\,dx \le C(\|P_0 u\| + \|u\|)^2 \qquad \forall u \in D(P_0).$$

Let $\chi \in C_0^\infty(\mathbb{R}^3)$ be

$$\chi = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 2. \end{cases}$$

Then, letting $E(x) = -1/(4\pi|x|)$ be the Newtonian potential in $\mathbb{R}^3$ (so $\Delta E = \delta_0$),

$$\chi u = \delta_0 * \chi u = \Delta E * \chi u = \underbrace{E}_{\in L^2_{\text{loc}}} * \underbrace{\Delta(\chi u)}_{\in L^2_{\text{compact}}}.$$

So $\chi u \in L^\infty$ is continuous, and

$$|\chi u(x)| = \left|\int E(y)\Delta(\chi u)(x-y)\,dy\right| \le \left(\int_K |E(y)|^2\,dy\right)^{1/2} \|\Delta(\chi u)\|_{L^2}.$$

Thus, $|\chi u(x)|^2 \le C\|\Delta(\chi u)\|_{L^2}^2$, so dividing by $|x|^2$ and integrating on both sides, we get

$$\begin{aligned} \int \frac{|\chi u(x)|^2}{|x|^2} &\le C\|\Delta(\chi u)\|_{L^2} \\ &\le C(\|\Delta u\|^2 + \|u\|^2 + \|\nabla u\|^2) \\ &\le C'(\|\Delta u\|^2 + \|u\|^2). \end{aligned}$$

1.2 Quadratic forms

Let H be a complex, separable Hilbert space, let $D \subseteq H$ be a linear subspace, and let $q : D \times D \to \mathbb{C}$ be a sesquilinear form. Let $q(u) := q(u, u)$ be the corresponding quadratic form with domain $D(q) = D$.

Remark 1.1. The polarization identity

$$q(u, v) = \frac{1}{4} \sum_{k=0}^{3} i^k q(u + i^k v)$$

allows us to determine $q(u, v)$ from $q(u)$.

Definition 1.1. We say that q is **symmetric** if $q(u, v) = \overline{q(v, u)}$ for all $(u, v) \in D$ (so $q(u) \in R$ for all u). A symmetric form q is **bounded below** if $q(u) \geq -C\|u\|^2$ for all $u \in D(q)$.

Example 1.2. Let $H = L^2(\mathbb{R}/2\pi\mathbb{Z})$, and let $V \in L^1(\mathbb{T}; \mathbb{R})$. Consider $q(u) = \int_{\mathbb{R}/2\pi\mathbb{Z}} (|u'|^2 + V|u|^2)\, dx$ with domain $D(q) = H^1(\mathbb{T}) \subseteq L^\infty(\mathbb{T})$. Formally, we can write

$$q(u) = \langle Pu, u \rangle_{L^2}, \qquad P = -\partial_x^2 + V.$$

We'll apply quadratic form techniques to discuss self-adjoint extensions.